**THE MATHEMATICAL AND PHILOSOPHICAL SIGNIFICANCE OF ZERO: NOUGHT MATTERS**

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**Abstract**

This is an exploration of the mathematical and philosophical significance of zero. It looks at the mathematics of zero, and the history of its development from three perspectives: syntactic (signs), semantic (meanings), and pragmatic (social use and contexts). It draws a clear distinction between nothing and zero, which avoids various paradoxes and contradictions. There is an analysis of the meaning of mathematical symbols, especially zero. After sketching various histories it entertains the hypothesis that a concept of the empty void is a necessary prerequisite for the development of proper zero (as a number not just a placeholder) in the history of numeration. This is confirmed positively in Indian and Mayan cultures and negatively in Ancient Egypt. Above all, it celebrates zero as essential in mathematics and digital culture. It credits Brahmagupta, at around CE 628, as the Indian analogue of Euclid with regard to integer arithmetic (Z), but repudiates the Great Human Theory of history which attributes advances to key individuals. It fails to resist making the odd joke about being, such as nought matters.

**Prologue**

If you wanted to schematise the history of numbering systems, you could say that it fills the space between One and Zero, the two concepts which have become the symbols of modern technological society. Nowadays we step with careless ease from Zero to One, so confident are we, thanks to computer scientists and our mathematical masters, that the Void always comes before the Unit. We never stop to think for a moment that in terms of time it is a huge step from the invention of the number ‘one’, the first of all numbers even in the chronological sense, to the invention of the number ‘zero’, the last major invention in the story of numbers. For in fact the whole history of humanity is spread out backwards between the time when it was realised that the void was ‘nothing’ and the time when the sense of ‘oneness’ first arose, as humans became aware of their individual solitude in the face of life and death, of the specificity of their species as distinct from other living beings, of the singularity of their selves as distinct from others, or of the difference of their sex as distinct from that of their partners. (Ifrah 2000, p. xviii)

This is a great piece of rhetoric and it identifies an important historical inversion. The normal order of progress, following the direction of increasing magnitude, goes from 0 to 1, from nothing to something. Instead, Ifrah asserts the reverse direction, from 1 to 0. Thus he inverts the universal order of the creation myth from nothing to something. But stop. This is of course a charming parable, a myth, a metaphor or a literary conceit, but it is nothing more. Although the inverted sequence parallels the course of historical development, it relies on false analogies.

Mathematical one, the number ‘one’, symbolised as ‘1’ has various meanings but these do not include the senses of ‘oneness’, unity, wholeness, ‘fullness’, individuality, solitude, specificity or singularity. These are merely analogues to identifying a first one for counting. The similitude is poetical, mythical, but not precise nor mathematical. Unity, wholeness, the sense of being one with the universe are marvellous and deep mystical feelings of connection, but are not within the subject matter of mathematics.

Likewise mathematical ‘zero’, the number zero, the sign ‘0’ also have various meanings but these do not include the void, the sense of ‘emptiness’, whether in the unformed or becoming universe or in the hollow or detached feelings of the individual much discussed within existentialism and mysticism. These can be found in Hinduism, Buddhism, Daoism, Sufism, Judaic cabbala, and doubtless many other intellectual traditions. But these are poetic, mythic or even mystic concepts, at the very best they are analogues for the empty set, but carry much more diffuse and extraneous meanings, far beyond the scope of mathematics. So they are qualitative analogues not the quantitative concept of zero. The empty set is of course the proper referent of zero in modern mathematics but this brings in the slippery and even more dangerous concept of set or class. This concept alone, confused with the collected referents of a predicative term, has been the ruin of more than one distinguished mathematician, including Gottlob Frege.

What this little excursion shows is that, even among the learned, there is confusion about the significance and meaning of mathematical terms, especially those terms with deep philosophical connotations such as zero, one, and let us never forget, infinity. Perhaps it is not ignorant confusion, but a deliberate attempt to amplify the significance of such concepts, to show how their echoes resonate throughout human cultures, evoking deep and lasting notes. This is all very well in a popular exposition more concerned with effect than accuracy, more concerned with the generation of excitement than the realization of precision. But mathematics, unlike poetry, must guard against the smuggling in of everyday meanings. These may well serve as epistemological obstacles, tripping up the more rigorous and austere chains of thought and reasoning upon which mathematics depends for its security. Indeed, such obstacles are encountered twice; first in the history of mathematics and then again in the development of the individual learner’s understanding of mathematics (Vygotsky 1978). Just as was the case in history, often a child will say “zero isn’t a number, you can’t count out zero sweets”. We have to overcome the myth that zero is simply the absence of number.

**Introduction**

Speaking of children’s understandings, I have few mathematical memories from my years of elementary education. But I do remember being perplexed. In my childish way of thinking I was puzzled by a coincidence. I noticed that the number 1 and the size of the step from 1 to 2 were both the same. And I thought – isn’t this a coincidence, that the first number and the step to the next should be the same size? One is a number and the other is a step, why should they be the same? Of course I didn’t then distinguish 1 as a cardinal whole number and 1 as a measure along the number line.

Of one thing I am confident. I had by then constructed a mental number line for myself, a private mental image of the linked counting numbers in their ever increasing size, but it was an idiosyncratic image (Ernest 1983a). It is closer to the irregular patterns of the ‘number forms’ noted by Galton (1883) among students and practitioners of mathematics, than the straight number lines that many construct for themselves today (Ernest 1983b). These are probably the result of working with graduated rulers. Indeed my own mental number line runs straight from left to right from 0 to 10, just like a ruler, before it wanders off, up and away, through the ‘teens, twenties, hundreds, and so on. It resembles a cable strung on poles receding up an hillside. When I discovered my mental number line in my 30s I realized it had long been present but unconsciously so, in my thinking of number. In uncovering this image and examining it introspectively I noticed that two non-significant number points stood out on the line, marked at 239,000 and 186,000. Rather odd numbers, not being rounded multiples or powers of ten). But then I recognised them: they represent the distance to the moon in miles and the speed of light in miles per second. In my boyhood, maybe 8 or 9 years old, I had a fascination with rockets, space travel, astronomy, and all the related ideas and wonderment. When I discovered my mental number line, thirty years later, the marks were still indelibly there, in my memory. This convinces me that I had already constructed my image during my primary school years and had highlighted these personally meaningful numbers during that phase of interest.

You will have noticed that zero is conspicuous by its absence from my little story, so far. But in fact it provides the key to unlocking the coincidence that puzzled me. In Peano’s (1889) axiomatic formulation of arithmetic he has 2 primitive notions or signs apart from those of logic. These are the numeral ‘1’ and the successor function which he designates as ‘+1’. Thus he views the beginnings of number in the same way as I did as a child, but of course, in his infinitely more learned and sophisticated way. But here you have the coincidence again, in the presence both of number 1 and the operation +1, which is also of size 1. Hence the paradox and my childish puzzlement.

Modern formulations of Peano arithmetic now use the numeral ‘0’ and the successor function designated as ‘S’ as primitives. This results in a system structurally identical to Peano’s original formulation, producing an equivalent set of natural numbers. However, it posits 0 as the first number and both of the derived numbers 1 (which is the successor of 0, S0) and 2 (which is S1 or SS0) as successors of preceding numbers. Likewise all further numbers are also successors of their predecessors. Thus the mystery is solved, the paradox of coincidence dissolves. 1 is S0 and 2 is S1, so of necessity the size of 1 (the step from 0) and the step from 1 to 2 are the same, for each represents a single application of the successor function S to the preceding number. One can say that 1 is the size of the step produced by a single application of this function. It creates both 1 and the size of the step to 2, and all subsequent numbers. We see therefore that zero is the only essential and ineliminable number, for 1 is derived from 0 as its successor.

Because natural numbers are used in the first instance as a set of ordinal counting numbers determining the first, second, third, and so on, in a sequence, it might be assumed that therefore the sequence is parametric, with no implications about the size of the steps of successive numbers. When numbers as used purely for enumerating an order, as in numbered paragraphs in a text, no assumptions about equal numerical spacing arise. However, the ordinality is soon absorbed into the cardinality function, which is the primary purpose of the natural numbers. Counting the members of a trio (one, two, three), for example, determines the cardinality of this set. Namely three, in the case of this trio. Once we are in the realm of cardinals, all step sizes produced by the successor function must be identical.[[1]](#footnote-1)

Notice zero (0) is conspicuous by its absence from Peano’s formulation, but its existence is necessary in the modern reformulation, in order to solve the mystery. This is the repeated story of zero. You can manage without it to a certain level, but to progress you must include it in the larger system, whether it be the formulation of numerals, the development of arithmetic, or the completion of algebraic structures such as groups, rings and fields. For zero, as well as becoming itself, the fully fledged concept of zero, through the long historical and epistemological development leading up to the idea, once it has appeared it is generalized to become the all-important identity element in many structures and theories. This identity element function is vital, and indeed I shall suggest that Brahmagupta’s decisive expansion of number to include zero is based on its identity properties with respect to addition. This follows on from the inclusion of negative numbers to form the integers Z, rather than in order to fill in empty place value columns or to represent the contents of an empty set (nothingness).

**The History and Nature of Counting, Numbers and Zero**

Counting, numbers and zero have been through a number of phases of development. Furthermore, there are several dimensions of counting and number to consider, including numerals (the signs), numbers (their meanings) and the social and historical contexts of the invention and use of arithmetic. To better incorporate each of these disparate dimensions I draw on Charles Morris’s (1945) tripartite division of the field of semiotics, the study of signs, into syntactics, semantics and pragmatics. I shall use this to structure my account. First, there is the technological development of counting, calculation and numeration systems, that is, the syntactical dimension. Second, there is the philosophical and conceptual development of numbers and number relationships, the semantic dimension. The third perspective, often neglected but equally important in my view is the pragmatic dimension. This represents the human, social and cultural needs, purposes and practices running through the societies and civilizations of the past five millennia within which the development of number takes place. Mathematics does not appear in a vacuum, self-generated, but arises as a result of complex human needs, and zero does not appear until much of the way through all of these intertwined histories, as Ifrah (2000) indicates.

*Syntactics: The Technological Development of Counting and Numeration Systems*

The technological development of counting, numeration and calculation like all other human technologies and knowledge structures, is driven by social needs and interests. In the course the history both numbers and numerals were invented and elaborated. Although there is an important distinction between numerals (signs) and numbers (the numerical values denoted by numerals) to avoid cumbersome expressions I will not distinguish between them except where it is significant and might lead to confusion.

Key stages or milestones can be identified in the technological development of counting, numeration systems and calculation. These are listed below in Table 1. This sequence or progression might be called the development of the syntax of counting and numeration. For it concerns the signs and their rules or grammar. I make no claim that this account is complete. The numbered stages listed below in Table 1 are approximately sequential in terms of historical emergence. But many are cumulative and are not superseded but remain functioning during the development of subsequent stages. Furthermore, it is more of an outline sequence of ideas than an accurate historical one.

Below, use is made of both Peirce’s (1931-58) and Bruner’s (1964) analyses of signs. Peirce makes the distinction between index, icon, and the purely symbolic. Bruner, himself influenced both by Peirce and Piaget, distinguishes enactive, iconic, and symbolic signs, where the last category includes spoken language, written alphabetic text, and written mathematical signs. These are explained below, following Table 1.

1. Verbal or oral counting, comprising spoken language signs
2. Tallying, the action of counting with successive strokes (enactive and embodied signs)
3. Tally marks representing the actions of counting as iconic signs
4. Numerals, incorporating tally marks for the first few numbers (iconic and symbolic sign mixes)
5. Numeral systems with different iconic signs designating different denominations (iconic and symbolic sign combinations)
6. Place differences with fixed ordering of denominational signs (e.g., Ancient Egyptian and Roman numeral notations, which retain stage 4 repetition of signs representing tallies)
7. Column notation representing place value, most likely derived from abacus or sand troughs used for accounting or calculation
8. Empty column notation – zero as a place holder
9. Place value notation, with zero as a sign representing nil value present
10. Place value notation, with zero as numeral of same status as other numerals
11. Decimal place value notation, with ‘point’ or other marker separating whole numerical values from fractional values
12. Admission of indefinite length fractional decimal notation
13. Admission of infinite length decimal notation
14. Invention of Binary number system with only two digits 0 and 1 corresponding to the states Off and On
15. Binary digital system used to encode computer programmes and data including numbers, text, pictures, video, music and all of human knowledge (all human knowledge products and representations)
16. Binary digital system used to encode all representable knowledge of humans (classificatory and control systems regulating human behaviour).

*Table 1: The Syntactic Development of Counting, Calculation and Numeration*

In the descriptions I use Bruner’s use of the terms enactive, iconic and symbolic signs. Enactive signs operate as gestures, such as pointing left or right, to give directions. As such they are a form of index signs as defined by Peirce, which indicate by proximity or adjacency. Enactive signs can also function as icons, representing meanings through movement, such as turning down fingers as one counts, or drawing a circle in the air with one hand motion. Iconic signs carry within themselves a structure resembling and partly constitutive of the meaning conveyed. Thus iconic signs employ metaphor or analogy to embody their meaning, at least in part. In modern times this has been applied in computer icons that carry little drawings indicating their function.

Symbols, the largest class of signs, are made up of arbitrary indicators associated with their meaning purely by convention. In Table 1, there is a degree of ambiguity in the interpretation of tallies. Tallying, the process of counting with successive movements or strokes is enactive as the actions correspond (in a one-to-one manner) with the set of objects that is being counted. Tally marks are however iconic, for the sign pictures the domain being counted through a one-to-one correspondence.

Table 1 indicates how zero plays a role of utmost significance in the development of the grammar and syntax of the sign systems of counting, numeration and calculation, hugely enlarging the scope of numerical applications.

Another dimension, which is woven into this historical story is the development of numbers themselves and number relationships, the semantics of number.

*The philosophical and conceptual development of numbers and number relationships (Semantics)*

A number of key stages in the conceptual and philosophical development of numbers and number relationships can be identified. These run in parallel to but behind the technological development of counting, calculation and numeration systems. They are listed in Table 2.

1. Development of numeration systems to quantify material collections of items
2. Extension of numeration systems so that non-present or imagined collections can be quantified, both as they are and after they have been configured and reconfigured.
3. Ontological concept of number as an independently existing entity and number mysticism (Pythagoras and his disciples)
4. Number whose meaning comprises ordinality, used in numbering sequential lists in a fixed order; cardinality representing the quantity of elements in discrete collections (such as the ‘twoness’ of a pair); and length as represented by the extension of geometric lines (Pythagoras).
5. The incommensurability of some lengths, which contrary to the Greek doctrine of commensurability and harmony, was known to Pythagoras and his disciples, applying to the side of a unit square and its diagonal (1 versus the square root of 2).
6. Zeno’s paradoxes that involve cutting up finite lengths to the infinitely small (length zero) which when reassembled (so it is argued) would be infinitely large. This involves the conceptualisation of lines with lengths approaching 0.
7. The recognition of fractional quantities resulting from division, first regarded as derived numbers, not given full ontological number status as mathematical objects. (Already anticipated, e.g., by the Ancient Egyptians)
8. The domain of pure number distinguished by Plato and his disciples from the technological ‘logistic’ of applied numeration in commerce, planning and governance. There is a concomitant higher valuation of pure conceptual number and number relations over the (perceived) inferiority of technological number, calculation and measurement (including numeration systems).
9. There are philosophies of emptiness, with nothingness conceptualised as a lack, a void, an absence, especially in Eastern philosophical and religious traditions such as Hinduism, Buddhism, Jainism and Daoism.
10. Conceptual shift from zero as negative sign for lack of quantity (non-existence) to positive sign for null quantity to be found within an empty collection. This is a shift from qualitative to quantitative meaning and understanding of zero, and is tied in with the previous stage.
11. Invention of signed numbers or integers with negative numbers -1, -2, -3, … on a par with positive or natural numbers 1, 2, 3. … (or +1, +2, +3,, … in modern notation)
12. Zero as number participating in numerical relations (e.g., +1 + -1 = 0) and representing identity under the operation of integer addition. This is tied in with stages 10 and 11 and represents the full concept of zero, identifiably a number treated on a par with other numbers in numerical relationships.
13. Special rules for all 4 operations (+, -, x, /) with 0, 1, positive and negative numbers, acknowledging that division by 0 is problematic (result undefined or infinite). This is tied in with stages 10, 11 and 12. The historical record attributes the systematic integration these stages to Brahmagupta (598–668 CE)
14. Decimal place value notation, with ‘point’ or other marker separating whole numerical values from fractional values
15. Admission of indefinite length approximations as series whose sums approaches a finite limit value, representing indefinite length fractional decimals
16. Admission of infinite series whose sum approaches a finite limit value and infinite length decimals thus representing an extended concept of number
17. Admission of limits with decreasing numbers in a sequence approaching 0
18. Introduction of infinitesimals, numbers different from but inseparably close to 0, in the foundations and workings of calculus (Newton, Leibniz) and its criticism (Berkeley). (infinitesimals are anticipated in the work of Bhaskara II in the 5th century CE, see Joseph 2018)
19. Founding of probability theory with 0 representing impossibility and 1 representing certainty
20. Recognition that zero serves as additive identity for the following sets of numbers: N (natural numbers), Z (integers), Q (rationals), algebraic numbers, R (real numbers) and C (complex numbers), and that it parallels one as multiplicative identity in these number systems.
21. The development of group theory with elements, operations, identity and inverses originating in the permutation of solutions in algebraic equations by Lagrange and others generalizing the familiar identity elements 0 (under addition) and 1 (under multiplication) in arithmetical systems
22. Generalisation of algebraic structure from integers, rationals, real numbers with operations analogous to arithmetical operations and identity element analogous to 0 (under addition) and 1 (under multiplication) in groups, fields and rings
23. The foundations of calculus are rectified with new epsilon-delta definitions of closeness to 0 and other limits (Cauchy), in 19th century.
24. Invention of Boolean algebra (laws of thought) with 0 as the lowest value representing falsehood and 1 as the highest value representing truth
25. Arithmetization of functions, calculation, logic and proof by Gödel and Turing, using primitive recursive operations, so that the signs are nothing but elements within rule based systems with no need of reference to meaning.
26. The automation of Turing’s ideal computing function as the basis for digital computing
27. The rehabilitation of infinitesimals in calculus in 20th century (Robinson).

*Table 2: The Semantic development of numbers and number relationships*

Table 2 lists many of the most important advances in mathematics from ancient to modern times, with 0 playing an important role in many of the stages. The table illustrates the stages through which that which is to become zero advances from non-existence via its negative conceptual precursors as nothingness, a lack, an absence, through positive concepts of emptiness, a pregnant void, an empty but nascent space in the process of becoming and bringing contents into being. As a result of this, nothing or the void is something, an empty collection in which the concept of the space or collection itself (an existent entity) is distinguished from the contents of that empty space or collection (non-existent, lacking objects, signs or anything). Finally, analogous with but distinct from this qualitative concept is the number zero itself, a fully-fledged number that reflects the cardinality of an empty collection, the number that immediately precedes one.

This is a jump or sidestep from the world of philosophical, mystical and qualitative concepts to the domain of properly quantitative mathematical concepts. The resultant number zero fully participates in numerical relations and, alongside positive and negative numbers, and lies within the range of possible answers to algebraic equations. This last characteristic is important, for with regard to existence in mathematics “to be is to be the value of a variable. More precisely, what one takes there to be are what one admits as values of one's bound variables” (Quine [1990](https://onlinelibrary.wiley.com/doi/full/10.1111/ejop.12534#ejop12534-bib-0029), p. 26). Thus when one accepts that solutions to equations may include zero or negative numbers one has acknowledged their existence, as possible and allowable values of a variable, x say. At this point, zero has truly arrived, it is the fully developed numerical concept with all of the properties and relationships needed to join the number systems N and Z. It can join the Pantheon of the other acknowledged-to-exist numbers including one, two, three, etc. Zero is a number. Zero exists on a par with all other numbers.

The semantic strand I am discussing in this section highlights mathematical concepts and objects, as well as meaningful operations and processes, comprising the main content of mathematics. In addition to its concepts the development of mathematics includes calculation methods, approximation methods of increasing accuracy for trigonometric functions and constants (such as pi and square roots), equation solving methods, and reasoning and proof methods. These are not treated explicitly in the present account, although the quadratic formula is mentioned in the history of Indian mathematics as admitting negative and zero values.

A further dimension also not treated here is the geometric meaning of 0 as the basis or origin from which lengths are measured. This is basis of measurements that record numbers such a 1, 2, 3, …, and so on, as length markings where point 0, implicitly or explicitly, precedes 1 by one unit of measure. It is further utilised in Descartes’ introduction of co-ordinate geometry in which the axes incorporate length markings, thus allowing the geometric representations of equations on the plane, or indeed in higher dimensional spaces.[[2]](#footnote-2)

*The social and cultural contexts and roles of number and numeration (Pragmatics)*

The third strand of development distinguished above concerns the social and cultural contexts of number and numeration; what might be termed the pragmatics of number. This is the most important but least treated element of the history and philosophy of mathematics, asking such questions as what social forces lead to the invention and development of number systems? What social purposes do number, calculation and indeed all of mathematics serve, both past and present? How is the social and cultural context woven in, providing limits, constraints, affordances and encouragements for the flourishing and development of mathematical practices?

Taking a pragmatic or externalist view of mathematics, where one takes social, cultural and personal factors into account immediately suggests a further comment on the admission of negative and zero solutions to equations, an admission that leads towards full numberhood. If you are investigating solutions to equations then there is a drive to accept negative and zero solutions that stems from the values of mathematicians. As we now know, and as they were finding out then, all quadratic equations point to two possible solutions. If those that are negative or zero are disallowed, then there is a displeasing asymmetry, a contingency factor in how you treat the equations. Admitting such solutions as legitimate permits a general method for solving all quadratic equations to stand. Indeed, versions of what we call the quadratic formula are present in Indian mathematics of the period. Mathematicians have always been susceptible to the drive towards generality, with the powerful attraction of achieving economy and elegance. These, indeed are partly what define mathematical beauty (Ernest 2015). Such a drive, coupled with the power and utility of the new methods, will likely help overcome the resistance, perhaps for some even the repugnance of letting zero and negative numbers cross the threshold into the legitimate and accepted domain of warranted proper numbers. However, such an admission, although warranted by a few leading-edge mathematicians, could only be the start of a broader acceptance which would take centuries to accomplish.

In order to be at least partly systematic, as in Tables 1 & 2, once again I propose a quasi-historical sequence for the development of mathematics from an externalist and pragmatic perspective. Since my main focus in this paper is on zero this is necessarily a truncated excerpt from the story that could be told, although even so restricted it remains extensive. However this list is too extended to be meaningfully made into a table.

*A listing of sequential events in the pragmatics of number development: The social and cultural contexts and roles of number and numeration*

1. Number systems were invented in ancient societies for the needs of taxes, kingly or priestly tributes, trade and accounting. The technological development of counting, numeration and calculation is driven by the needs of trade, tax and tributes, that is to record and warrant transactions and license and legitimate the quantities of goods involved, including calculating arable land areas.[[3]](#footnote-3)
2. These primary social functions necessitate an underlying structure and certain basic requirements. Namely, above all else counting, quantity recording and arithmetical systems must be permanent, unambiguous, and invariant under repeated counts and under operations. But before we start counting there must first be agreement as to what constitutes a unit, for example a goat, basket of grain, or an ingot of metal. This depends on the concepts of unitizing, equivalence and replicability. There must also be agreement on a fixed numeral sequence (oral or recorded), to enable determinate counting procedures and constant results. The outcome is the permanence of counts, as the unvarying cardinal number representing the numerical value of a designated collection.
3. Furthermore, physical combinations or partitions of goods (already invariant under the principle of the conservation of matter) must be mirrored in the invariance of quantity as represented by number under calculative operations.
   1. Addition represents combined regroupings, putting collections of goods together
   2. Subtraction represents partitioning or removal of goods
   3. Division represents repeated partitioning of collections of goods, with fair sharing
   4. Multiplication represents repeated regrouping of goods, with collections of equal size combined together

All of these operations involving the partitioning or conjunction of collections of goods must be reversible, just as the corresponding physical actions are. Overall, numerical values must be preserved in both directions. Thus, for example, 3 baskets of grain combined with 5 baskets will always total 8 baskets. 8 baskets with 3 baskets removed will always result in 5 baskets. Putting it more abstractly in modern symbolism; on the basis that 3+5=8, we must always be able to assert that 8=3+5, 8=5+3, 8-3=5, etc. To the modern eye, these numerical equations appear to be trivial, but they embody the invariance and reversibility of number operations that are essential in the realm of calculation for it to fulfil its social function. I must emphasise, contrary to the widespread belief, that it is not that arithmetic by its very nature imposes these constancies. Rather it is that the social demand for an invariant system of accounting requires that arithmetic embodies these properties. However, this character has been so reinforced through mathematical custom that invariance and reversibility are now understood to be intrinsic properties of number and calculation, rather than the outcomes of societal constraints and demands on systems of numeration.

1. Numerical systems evolved throughout history so as to be able to encompass large numbers, and to permit efficient and succinct forms of record keeping. Calculation developed so as to meet the needs of financial applications as well as other problems arising from measurements and mensuration, building, astronomy, and so on. In addition, there are also theoretical problems posed by scribes, tutors and mathematicians and both designed for educational uses, as well as puzzles for the exercise of their own virtuosity as mathematical practitioners. Such activities from within mathematics result in conceptual extensions and developments that are internally generated (that is, internal to the social practices of mathematics).
2. There are philosophies of emptiness that originate in many overlapping domains of theology, mythology, mysticism and philosophy across cultures and religions including Hinduism, Jainism, Buddhism, Taoism, Ancient Greeks, Sufism, Judaism and Cabala, and Christianity over several millennia. In these belief systems there are philosophies of emptiness; nothingness is conceptualised as a lack, a void, nil, nothing, an absence. But these conceptions mean that although nothingness denotes a void, nothingness exists as a concept, so ‘nothingness’ is something, namely the name of nothingness. It is almost certain, that such conceptions are necessary prerequisites for the development of zero into a number concept. Nothing becomes something. It gains ontological status as something, even if it is devoid of content. This is perhaps necessary, although certainly not sufficient, for zero to emerge.
3. In the centuries preceding the 7th (CE), sometime after second century BCE negative numbers were used in India to represent debts, and positive numbers to indicate assets or ‘fortunes’[[4]](#footnote-4) (Mattessich 1998). Thus a lack of debt could be conceptualised as something real, alongside possession of something, which whether substantial or not, is real and countable. An economically and fiscally advanced culture was almost certainly necessary for the development of negative numbers as numbers on a par with natural (or positive) numbers. However, this involves a significant conceptual step beyond numbers as representing something materially present to the senses, or potentially so. For quantified ownership can be verified individually by the senses, whereas financial assets or debts can only be determined by reference to social agreements and documents recording them. Thus debts as negative numbers represents a significant conceptual advance in the development of number systems.
4. Brahmagupta (598–668 CE) clearly was familiar with negative numbers when he articulated the properties of the numeral zero as a number in its own right. He defined it as the number you get when you subtract a number from itself.[[5]](#footnote-5) He thus made (or employed) the further step in abstracting numbers from the results of counting objects materially present to the senses. Beyond having the concepts of debt and of emptiness it is still a giant step to conceptualise zero as a number on a par with other numbers. The third and most important conceptual foundation for this development is the conception of number and operations as comprising a totality; an interconnected system of relationships and meanings.

It is evident that Brahmagupta held this view because he also defined rules for all four operations with the number zero, as well as for combining signed numbers (debts and assets). Thus he accepted that both 2-3 and 2-2 are legitimate operations and that each provides a recognisable number as a solution (-1 and 0, respectively). He created an elaborate theory including all of these components, thus more or less establishing the modern domain of integers (Z) as it stands today. Brahmagupta also worked in algebra and regarded zero and negative solutions as acceptable and legitimate. Indeed, as I argue above, working with equations and their solutions provides a further impetus for acknowledging and treating zero and negatives as proper numbers.

By bringing together and extending what was known about positive and negative numbers and zero and all of their relationships, Brahmagupta was making a great synthesis and a giant leap forward. Brahmagupta is in many senses the Indian analogue of Euclid, as Table 3 shows.

|  |  |  |
| --- | --- | --- |
|  | **Euclid** | **Brahmagupta** |
| *Domain formalised* | Geometry | Arithmetic |
| *Epistemological basis* | Knowledge as based on deductive reasoning, sourced from philosophy | Knowledge as based on lists of rules defining legitimate utterances, sourced from the grammar and linguistic theory of Panini |
| *Theory created* | Deductive plane Geometry | Domain of integers and its rules for arithmetical calculation |
| *Objects* | Constructible plane figures | Integer numbers including zero and negative numbers |
| *Rules of the theory* | Rules for determining congruence, rules for construction | Rules for combining positive integers, negative integers, and zero using the four arithmetic operations (+, -, x, /) in calculations |
| *Additional contributions* | Books of the *Elements* on solid figures and number | Significant contributions to mathematics of astronomy, trigonometry and well as algebra and the solution of equations (including positive, negative and zero solutions) |
| *Incorrect attribution* | Inventor of deductive geometry. *Instead Euclid is in fact the systematiser of geometric knowledge .* | Inventor of zero and integer arithmetic. *Instead Brahmagupta is very likely the systematiser of an arithmetic tradition going back to Aryabhata and earlier* |

*Table 3: The analogy between the contributions of Euclid and Brahmagupta[[6]](#footnote-6)*

Table 3 makes explicit the analogy between Euclid’s and Brahmagupta‘s contributions, and it is quite striking. Many dimensions tally. However, despite the strength of the analogy, there are notable differences too. Euclid’s *Elements* has an elaborate logical structure in which the ordering is crucial, for later theorems depend logically on earlier ones. Brahmagupta’s text on arithmetic has no such structure. The order of the rules for the most part does not matter, just as the rules of grammar are not rigorously sequenced either. Another difference is that Euclid’s *Elements* has little application other than in education and philosophy. In contrast Brahmagupta‘s rules impacted on algebraic practices and on the regularisation of number and numerals. Subsequent developments in extended decimals, trigonometry and working with infinite series would not have been possible without this contribution. Several other practices such as astronomy also depended heavily on these numerical innovations.

1. The word zero retains clear signs of its Hindu and Arabic roots. The Indian name for zero was Sunya, meaning ‘empty’. When the Arabs adopted Hindu-Arabic numerals, they also adopted zero the term for which they turned into ‘sifr’. Some Western scholars turned sifr into a Latin-sounding word, ‘zephirus’, which is the root of the word zero.[[7]](#footnote-7) Other Western mathematicians termed zero ‘cifra’, which became ‘cipher’. Because of the import of zero for the new set of numbers, people started calling all numbers ciphers. This gave the French their term ‘chiffre’, digit, as well as the modern English-speaking world their name for code, ‘cipher’ (Seife 2000).
2. According to Seife (2000) and others, neither zero nor negative numbers were acceptable in Europe for several hundred years. He argues that the acknowledging the void, as acceptance of zero implies, challenges Aristotle’s doctrines and the beliefs of Christianity. It was only in the 12th century that these doctrines were rejected and not long afterwards Leonardo of Pisa (Fibonacci) introduced zero and negative numbers in his book *Liber Abaci*, published in 1202. The lesser known Nemorarius also introduced zero to Europe (Joseph 2008). Fibonacci was thus among the first two Europeans to accept zero and negative numbers as permitted solutions to quadratic equations. Like Brahmagupta, Fibonacci interpreted negatives as debit quantities.
3. Acceptance of Hindu-Arabic numerals, the related methods of computation and new additions like zero and negative numbers took several hundred years and as is well known the traditional abacists (using the abacus) were trusted more than the new-fangled algorists (who used the new algorithms and zero) in trade and business.
4. As I have indicated, the subsequent development of modern mathematics involves infinitesimals, limits, probability (0 represents impossibility), Boolean algebra (0 represents falsehood), infinity (Cantor) and the axiomatisation of arithmetic (Peano) and the representation of most higher mathematics within arithmetical systems (Gödel).
5. Focusing on binary alone, this is a numeration system that uses only the signs 0 and 1 in digital computing and the encoding of all information and representable human knowledge. It is astonishing to think that almost all of human knowledge and information can be encoded in binary using only 0 and 1 and that it can be compressed onto memory sticks and hard drives so that I can probably hold the contents of every book ever written in the palm of one hand. Wi-Fi, 5th generation mobile phone signals and optic fibre cables transmit live data streams that encode thousands of movies, millions of audio tracks and trillions of pieces of data, at the very least.[[8]](#footnote-8) Such data is both accessed and read by us, but also encodes our actions and behaviours in communicating (e.g., email, mobile phones, social media, Zoom, Teams) expressing our opinions, web-surfing, shopping, looking up locations, accessing services and other items and events that interest us. Our credit and loyalty cards, mobile phones, Fitbits, tablets, and other electronic devices reveal our physical locations, contacts, bodily movements, measures of health, consumption preferences and so on. The data we knowingly and unknowingly disclose is harvested and used to model and record our online behaviours and preferences, personality traits, beliefs, tastes, friendship links, dating behaviours, political inclinations, and the emotive issues and memes we react to, our ‘hot buttons’ (Merriam-Webster 2020). This data is used as basis for surveillance capitalism, the surveillance state, predictive and ‘nudge’ politics, with limited, if any, attention and restraint with regard to privacy and ethical boundaries (Ernest 2018). Who could have anticipated that 0 and 1 could be used with such devastating power and effect (including the extensive benefits not mentioned here)?
6. Digital systems and Apps are not only used to survey and nudge us towards commercial decisions and purchases. All this captured data is also analysed, modelled and used performatively to shape, construct and transform our views of social institutions, our views of reality, our views of self and our own identity; and consequently, our behaviours.

The perspective that the pragmatics of zero, and of mathematics more widely affords, enables us to evaluate the social role and function of mathematics. How in the third millennium it is inescapably involved in shaping our lives and our selves. The traditional perception of the ethical neutrality of mathematics is a hurdle that must be overcome in order to recognise the power that mathematics and the digitisation of the modern world has over us. As citizens living in a modern democracy we have the right to know about the intrusions of digital information systems into our lives, to call for transparency, accountability and ethical boundaries in these surveillance and shaping systems, and to bring some to a halt. At the same time as humanity has sleepwalked into an environmental disaster by allowing our world to be polluted with plastics, chemicals, carbon dioxide and other poisons, we have also allowed our selves, society and cultures to be digitally polluted, manipulated and moulded to serve forces beyond our consent.

The mathematics of programmes and Apps is performative, changing the way the world is categorised, transforming the way that public and corporate policies are enacted across all of the systems of justice, health, education, benefits and governance, banking and finance. But the forces that are surveilling and shaping us do not always have our best interests at heart. They are driven by the mission to control us in every sphere of our being. This is to make us socially compliant and well-functioning centres of consumption and labour, thus driving up corporate profits and reducing questioning of, and challenges to, the status quo. These forces are also attempting to manipulate us to support populist individuals and movements on the political stage. Who would have thought, who could have imagined in their wildest dreams, that the invention of the number zero and its utilization in binary coding could have led to this state of affairs? Zero may not only be the most significant conceptual addition to our systems of thought throughout the history of human culture and civilization. It may also be the genie in the bottle, the sleeping danger in Pandora’s box that we have unleashed that will haunt and trouble us forever.

Adopting the perspective of the pragmatics of zero; seeing how it is enmeshed in our history and socio-cultural contexts, opens up a broader and deeper history of the forces that shaped mathematics and zero and brought them into being. It also opens up new vistas on the roles of zero and mathematics in our lives, showing how powerful is their impact. It leads to questions about the legitimacy of focussing only on the internal dimensions of mathematics and the necessity of adopting a broader view that raises questions of control, power, politics and ethics, areas traditionally excluded from mathematics and its philosophy. But we should never forget it was the interests in trade, tax, tribute, calendar, astrology and astronomy that were the forces that brought mathematics into being and supported its development. Making the calculation systems more efficient for these purposes is what led to the invention of zero. Thus despite the ideology of purism adopted first by the Ancient Greeks, and then again by professional mathematicians in modern times, mathematics cannot and should not be viewed divorced from its internal and external histories and contexts (Ernest 2020). To do so is to seriously misrepresent and distort it.

**How are the Syntactics, Semantics and Pragmatics of Zero and Mathematics Valued?**

I have told three stories about the historical and philosophical development of zero. Here I consider the philosophical significance of each, although I already strayed into philosophical and ethical considerations arising from the employment of base two numbers, that is binary coding based on 1 and 0, in modern information, communication and control technologies.

*The Syntactics of Mathematics*

The syntactical story is about the sign-system of mathematics and when zero emerged and was incorporated into mathematical notation. Histories of mathematics will discuss and trace the various numeral systems and their development. Typically they do not explore the actual reasons why such sign systems were developed, but assume they were needed for counting and calculation, without inquiring what particular social needs lay behind this technological innovation. However, some of the best modern scholarship, such as Høyrup (1994, 2002) and Yadav and Mohan (2011) are meticulous in plotting both the detail of historical numeration systems, their mechanics and their social and historical contexts and usages.

Nevertheless, historians of mathematics from the 19th century on often subscribe to the Eurocentric ideology that attributes the decisive conceptual innovations in mathematics to the Ancient Greeks (Bernal 1987). This ideology interprets the contributions and advances in Ancient Egypt, Mesopotamia, India and China as either minor or as having come from elaborations of Ancient Greek ideas. Thus, the English orientalist, Henry Thomas Colebrooke, author of the Sanskrit Grammar (1805), undertook the task of translating classics of Indian mathematics, including the *Brahmasphuta Siddhanta* of Brahmagupta (Heeffer 2011). This showed that there existed an Indian tradition in which algebraic problems were solved with multiple unknowns, in which zero and negative quantities were accepted as fully fledged numbers, as well as other innovations. In explaining this tradition, which occurred during a period when mathematics was hardly practised in Europe and in what was (or was to become) the Islamic regions, scholars are divided into two opposing camps. Heeffer (2011) calls them the believers and the nonbelievers.

The believers showed an admiration for the ingenuity and originality of the Indian tradition. However, nonbelievers, did not grant Indian mathematicians the status of original thinkers. They argued that Indian knowledge must have been passed on from the Greeks, itself the cradle of Western mathematics and civilisation. A major and influential nonbeliever was Moritz Cantor (1880), who published a four-volume work *Lectures on the History of Mathematics* over the period 1880–1908. This is regarded as the first modern and comprehensive history of the subject (Swetz 2016), but it derogates the Indian contributions to mathematics. Cantor “takes every opportunity to point out the Greek influences on Hindu” mathematics, whether imagined, coincidental or real. He thus denies the Indians any credit for originality of thought or for what we can see today are major advances in mathematics (Heeffer 2011, p. 138). This is a manifestation of what we term Eurocentrism.

In the present context, the Eurocentric ideology is comprised of two strands. These concern the questions of who can do mathematics, and what is mathematics? The Eurocentric answer to the question of who can do mathematics is frankly racist, in that it regards the peoples of Ancient Egypt, Mesopotamia, India, China and elsewhere (including non-Europeans in Africa and the Americas) as intellectually inferior, capable only of copying but not of originating ideas in mathematics, science and other domains of knowledge. It is no coincidence that Eurocentrism was at its height during the most financially successful years of the British Empire when India and other colonies were being exploited and looted. Eurocentrism supports the imperial project through promoting European (English primarily) exceptionalism and superiority, which justified the exploitation of what were viewed as ‘inferior peoples’. The ‘fact’ that the minds of these subaltern peoples were incapable of the highest levels of thought confirmed that they were inferior, and not deserving of the rights and respectful treatment that is the due of Europeans, and especially, English gentlefolk.

The idea that humans and their cultures progress through distinct stages development is an ancient one, and already in the 16th century Montaigne (1580) criticized the assumption that civilized humanity were in some sense better than savage or barbaric peoples. Although the terms were ill-defined, in general meaning savages were tribal peoples, and barbarians were non-Christians that lived in organized societies. In the 19th Century Darwin’s theory of evolution reinforced ideas of hierarchy and development as you ‘ascend’ among all life forms, and especially humans. “Victorian-era anthropologists widely accepted the terms savagery, barbarism, and civilization to describe the perceived progression of human society from the most primitive forms to the most advanced. Such distinctions, however, reinforce xenophobic views used throughout history to justify hostile acts toward dissimilar cultures.” ([McNeill](https://1lib.uk/g/William%20H.%20McNeill), et al. 2010, p. 568). The idea that savage or barbaric peoples could be well advanced beyond Europeans, as Indian mathematicians undoubtedly were in what Christians call the dark ages, and Islamic mathematical culture was in medieval times, was not only anathema but inconceivable to the Victorians and indeed throughout much of the 20th century.

The Eurocentric answer to the question of ‘what is mathematics?’ attributes the decisive conceptual innovations in mathematics to the Ancient Greeks. This ideology is one that regards the geometry and pure mathematics of the Ancient Greeks, centred on proofs and reasoning as superior, and indeed as the only ‘true’ or ‘real’ mathematics (Ernest 2009). “Oriental mathematics may be an interesting curiosity, but [*only*] Greek mathematics is the real thing.” (Hardy 1940, p. 12, my italicised addition in brackets). The decisive Indian and Arabic work on number, algebra, trigonometry and analysis, which constitutes a huge leap forward from ancient mathematics, is brushed aside. Only wilful ideological blindness can possibly account for this great lacuna in the history of mathematics.

It is no accident that the ideology of purism reinforces the Eurocentric perspective on mathematics. Purism values pure proof-based mathematics as significant epistemologically, pertaining to truth, wisdom, high-mindedness and the transcendent dimensions of being. Applied mathematics and calculation are denigrated as technical and mechanical, pertaining to the utilitarian, practical, applied, and mundane; understood as the lowly dimensions of existence. Pure mathematics is the domain of philosophers, free thinkers and gentlefolk. Applied mathematics and practical arithmetic, termed logistic by the Ancient Greeks is the domain of lower class and trades people, or even, in ancient times, slaves (Plato 1941).

These Eurocentric and purist values have likely contributed to the scholarly separation of the history of its terminology, which I have termed here the syntactic, from the history of mathematics proper (Høyrup 1996, p. 7). One striking example concerns prime decomposition of numbers. The Prime Factorization Theorem asserts that every number n is the unique product of primes, and can be represented thus:

*Formula 1: The prime decomposition of number n*

In formula 1, the letters *i, k, n, c* and *p* all stand for natural numbers, *pi* is the ith prime number in order of size, starting with 2. Every natural number *n* can be uniquely represented this way, and any product in this form represents a unique natural number. The proof of these facts is regarded as one of the great achievements of number theory.

However there is no significance whatever accorded to a parallel result, that might be termed the unique numerical representation theorem (in any base). In decimal form this asserts that any number *n* is the unique sum of powers of ten:

*Formula 2: The decimal representation of number n*

In formula 2, *i, d, n* and *c* are all natural numbers, 0 ≤ *di* ≤ <9, that is, *di* is one of the ten digits. Every natural number *n* can be uniquely represented this way, and any sum in this form is a unique natural number.

The analogy between these two forms is as striking as is the difference in their respective valuations. Notice also that it would be difficult to manage without the sign 0, as *pi*0 = 1 plays an important part in Formula 1, and 0 plays an important part in the values of *i* (one of the ten fundamental digits), in the symbol 10, as a zero value in the overall sum when *di* = 0, in Formula 2.

There are two contingent factors in this decimal representation (formula 2), which make it less elegant and universal than the prime factorisation theorem (formula 1). These are the presence of the base number 10, and the restriction of *di* to values less than this base number. Of course this sum can be generalized to any base *b* provided we restrict the numerals *di*, to 0 ≤ *di* < *b*. But the contingency in the choice of some base factor cannot be eliminated.

This unique numerical representation is used universally and is displayed as the concatenation of digits *di di-1 … d2 d1 d0* for an i+1 place decimal (whole number) numeral, with the option of displaying a ‘point’ (.) at the right hand end to separate the digits such as these, representing whole numbers, from additional ones representing fractions.

Thus formula 2 allows for the unique representation of any natural number in a form that facilitates computation for the natural numbers and the integers. This is vitally useful and important, and really it is this that justifies the invention of numbers. Rather, it is the purpose for which numbers where invented, namely an arithmetic of invariant quantities. In contrast formula 1 offers no aid to computation, although it does underpin certain forms of cryptography and the completely theoretical Godel (1931) numbering.

However, beyond the representation of natural numbers and integers formula 2 allows for the extension of these number systems to include the rational and real numbers. This is of inestimable value and importance for most branches and applications of mathematics. The extended formulation is shown in Formula 3.

*Formula 3: The decimal representation of real number r*

In formula 3, *i, d* and *c* are all natural numbers, 0 ≤ *di* ≤ <9.

This might be termed the real number representation theorem or definition in base 10. This representation is unique, except for the recurring nines problem. (Given *dk+1*> 0 such that *di* = 0 for all values of *i* < *-k*, all of these values of *di* can be replaced by 9 provided *dk+1*is replaced by *dk+1*-1.) For when the fractional value digits do not end in a line of zeros, what we have is an infinite series whose value is the limit of the sum. Thus, for example 0.9999…. (recurring) = 1.

What is it then, that makes formula 1 so much more valued and admired than the far more practically significant and useful formula 2?[[9]](#footnote-9) The second is in use or assumed millions or billions of times daily, albeit implicitly, as representing the underlying meaning of decimal numbers (numerals). Formula 1 is an elegant piece of pure of mathematics used in number theory, whereas, in theory, formulas 2 and 3 are only of practical use in the application of number and mathematics. You will see decimal numbers throughout research papers in pure mathematics. However, they are largely regarded as conceptually insignificant, involved only in the ‘book-keeping’ or syntactics of mathematics.

Furthermore, once we include the extension to formula 3 it would be difficult to work in probability, statistics, trigonometry, mensuration without it, and all of modern calculus and analysis would be impossible. All of these branches of mathematics rely on the representation of real numbers given by formula 3 and would simply not be possible, at least not in the terms in which we understand and employ them today.

All in all, my claim is that the syntactical dimensions of mathematics, evidenced in the symbols systems of mathematics, and especially in the decimal (or other base) place value system, are regarded as trivial, mathematically uninteresting, devoid of significant conceptual content. This is despite the fact that what I estimate to be over 90% of the efforts expended in the history of mathematics have been involved in the invention and development of numerical representations and numerical algorithms, and well over 99% of human uses of mathematics concern nothing more than number representation systems and calculations.

What this disparity of treatment and regard shows is that there is an underlying and too rarely discussed value system underlying mathematics and its history. My contention is that it is no accident that it serves the interests of power elites and hierarchies in society. These favoured the free and privileged citizenry in Ancient Greece, members of a leisure class, over the humble tradespersons and slaves and their professional practices. In modern Europe these values favoured the upper and middle classes (gentlefolk) over tradespersons, workers, and this racist perspective regards as lowest of all, aliens from colonies, conquered nations and other non-European origins. The ideology of purism cannot be separated from issues of conquest, colonialism and empire and with its concomitant Eurocentrism and frankly racist views of subaltern and conquered peoples and their cultures as inferior. Despite the myth of neutrality the judgements of value and worth widespread within the traditional history and philosophy of mathematics are distortions, far removed from objectivity and fairness (Powell and Frankenstein 1997, Joseph 2018)

*The Semantics of Mathematics*

The development of the concepts, meanings and conceptual structures that make up the mathematics of zero are traced above, in Table 2. While far from complete as an history of mathematics or even an history of all of the mathematics conceptually related to zero it nevertheless shows how deeply zero permeates so many of the concepts, meanings, relationships and theories of mathematics. It shows first of all, how complex and elaborate the concept of positive number is before we even start to consider zero.

From a semantic perspective the big question is: what are the meanings of mathematical signs and symbols? The answer to this question is controversial. The traditional response is that a mathematical sign denotes a mathematical object or procedure. Thus 2 denotes the concept of Twoness, an abstract concept that can be applied to any pair of objects considered as a collection in itself. Likewise 0 denotes the number Zero that stands for the value of any empty numerical descriptor such as the number of dogs currently orbiting the Earth, or the cardinality of any empty collection such as the set of real roots of the equation x2 = -1. More generally, 0 is cardinality of any empty set.

Similarly, + is a binary operation of addition defined on the class of natural numbers. Formally, it is defined inductively, rather than explicitly. Thus, for any natural number *n*,

*n* + 0 =def *n*,

*m*+(*n*+1) =def (*m*+*n*)+1.

Here ‘=def’ means ‘is defined to be’ (specified to be identical in numerical value to), which is a binary relation in the metalanguage of arithmetic. Thus *m*+*n* is a computable operation that can be completed in *n* steps, with *m*+*n* reducing to SSS…S*m*, in which there are *n* applications of the successor function S. As this example illustrates, the procedures of mathematics are well defined operations on the objects of mathematics. However, such procedures can also themselves be viewed as objects of mathematics, so a cumulative process of definitions results in a growing collection of mathematical objects and procedures of ever increasing complexity and abstraction. In addition, the term *n* is a variable in the metalanguage of arithmetic in which its operations are defined.

The variable *n* ranges over the set of natural numbers. (Indeed. the class of natural numbers N is also defined inductively as follows: 0∈N, and if *n*∈N, then S*n*∈N.) But an arbitrary *n* must in fact itself be represented by a complex term, shown in general form by Formula 2, with the parameters given specific values. If we did not have place-value representations of numbers we would need an unlimited number of different numerals. Thus, for example, in Roman numerals we have distinct signs for 1 (I), 5 (V), 10 (X), 50 (L), 100 (C), D (500). 1000 (M), taking us to just beyond 1000 and we would need a distinct sign for each power of ten beyond these. Roman numeration also uses the convention that the sign for 4 is 5-1 denoted, through positioning, as IV (a number denoting letter on left hand side of the next higher denomination letter is negative), likewise for 9 (IX), 40 (XL). 90 (XC). 400 (CD), 900 (XM). Thus 107 is CVII, and the highest number expressible in Roman numerals is 3999 (MMMCMXCIX). To express 1,000,007 or any larger number is impossible without breaching the convention or without the addition of additional digits.

In base ten, 107 is the numeral standing for and thus abbreviating the compound term 102x1 + 101x0 + 100x7, and likewise 1,000,007 is 106x1 + 100x7 (with the zero terms omitted). In the representation of 107 as a sum of powers of 10 the brackets are omitted because of the associativity of addition. The property of associativity means that the value is demonstrably invariant no matter how the parentheses are inserted. However, such is the power of human understanding that the abbreviated numeral is a sign in its own right, not needing to be unpacked or reduced to its components except when operations are performed on it, such as adding it to another number also in this decimal form. Even then we have abbreviated procedures that mean that it does not have to be fully unpacked. As millions of schoolchildren learn annually, the sum 107+107 can be performed as follows:

1 0 7

1 0 7 +

2 0 14

2 01 4

2 1 4

Here each column is treated by the following sub-algorithm starting with the right-most (lowest denomination).

1. In any column, given digit *n* over digit *m* look up or compute the value of *n*+*m* = *k*, say.
2. Check the column’s memory box for contents 0 or 1. If the content of 1 is found redefine *k* to be *k’* = *k*+1 and use *k’* in place of *k*. Insert the value *k* in the next (lower) row of the column, provisionally.
   1. If *k* <10, confirm the value *k* in row 3 and move on to the next column on the left and repeat the procedure. If the last column has been treated stop.
   2. If not-(*k* <10), then *k* = 10 + *j* where *j*<10. In this case cancel *k* and leave just *j* in the third row and put 1 in the memory box associated with the next row.
3. Now repeat with the next row (on the left).

Such procedures are of no theoretical interest to mathematicians, even though they may employ them themselves, but are typical of the computation skills taught in schools to children. Such algorithms, determinate procedures with a unique output, are of course easily automated and are performed on hand held electronic calculators or computers. These computations are performed without recourse to the full expansion of numbers into the basic form shown in Formula 2. However the rules employed are the minimum required to respect the meaning of this underlying form. Thus when the sum of two digits in a column exceeds a single digit value, the ten part created in the two digit sum is recorded in the next larger value column (to the left) as a single unit that is incorporated into its two digit sum. For the local sum in row *i* of denomination 10i cannot be represented by a digit *k*≥10, and so it is broken down into 10 + *j* of value 10i+1 x1 + 10ix*j*.

Psychologically, the phenomenon of treating a compound sign as a single entity in its own right is known as chunking. A number (numeral in fact) such as 107 is seen as a single entity, and only needs to be opened up and examined in terms of the relationships between its constituent signs for certain purposes. Likewise, the student of chemistry will see the formula C2H5OH as a single entity, namely the formula for ethyl alcohol. Chunking frees up the mind to understand and process complex signs as single entities, enabling more and more complex signs to be treated as single objects of attention. This corresponds to our definition of the end product of mathematical processes applied to several mathematical objects as unitary mathematical object in its own right.

Another effect has been remarked upon in the linguistic analysis of number. Numerals and number words “do not refer to numbers, they *serve as* numbers” (Wiese 2003, 5, original emphasis). This is an important point that contradicts the simplest referential theory of meaning, the picture theory of meaning. Numerals, number word terms, and by extension all mathematical signs need not indicate or refer beyond themselves to other objects as their meaning. They themselves serve as their own objects of meaning. Mathematical language is performative, for mathematical terms create the objects to which they refer.[[10]](#footnote-10) Counting creates numbers, and operations create functions. In the first instance these are inscribed numerals, and enacted operations. Repeated usage reifies and solidifies them into mathematical objects. Their currency serves as a social warrant for them, verifying their robust and legitimate existence. This is a point of deep ontological significance, for we do not have to posit a realm of being in which to locate mathematical objects outside of space, time and material existence, the position of Platonism. Their use in social practices is what makes mathematical objects real. However, this is not to dismiss the problems of ontology, meaning and semiotics from mathematics, it is just that no special attention needs to be paid to these characteristics of mathematical signs when they are in use. Their meanings are self-contained. Thus it suggests that, for the most part, mathematical practices can proceed without worrying about ontology.

Historically, zero first served as a sign for an empty column, without being a number itself. Later in the development of zero it became understood to be a fully-fledged number in itself, Ironically, because mathematical signs can be used instrumentally or mechanically, it means that zero can once again be used without reference to its meaning. However, in this third stage of utilisation, the meaning of zero can be ignored in exactly the same way as those of all other numbers, so it remains a fully-fledged and indeed a rather important and distinguished number, like the rest. There is one difference. For example, in unpacking 1,000,007 as 106x1 + 100x7, it is clear I simplified. As I remarked, I omitted the terms 105x0 + … +101x0, since each has the value 0 and can be ignored in additive expressions as they do not affect the overall numerical value. Zero has a unique and distinguished place as the identity element of the natural numbers with respect to addition. It is the only number that can be ignored in addition.

Returning to the issue of meaning, it must be said that it is of great importance for mathematical signs, even if it can be temporarily ignored in some contexts. I have argued that for several reasons the referential theory of meaning fails for mathematics. Since the mid twentieth century the Meaning as Use theory has become prominent, drawing on the work of Wittgenstein (1953). Wittgenstein offers a whole range of insights into meaning including Meaning as Physiognomy, whereby meaning can be given by the shape or inner structure and arrangement of the parts of the sign. This clearly applies to, and offers insight into, the signs of mathematics. It parallels Peirce’s conception of iconic signs, whereby the inner structure of a sign in some sense is an analogue of that which it signifies, representing it pictorially or by some other means. For example, the Roman numeral III represents Threeness by corresponding to three tally marks. The numeral utilises a materially given instance of Threeness to designate 3.

Notice that this iconic conception of meaning gives rise, at least in part, to the historical and conceptual problems of accepting zero as a number. If one stroke (I) represents 1, two strokes (II) represent 2, three strokes (III) represents 3, then zero or no strokes () represents 0. But that means there is no sign for zero. Therefore, zero is not a number for all numbers have their signs. In this account I use parentheses to enclose and thus indicate the numeral as given by strokes. Otherwise, there would be only an empty space in the sentence where the numeral for zero should be located. This is a recurring and fundamental problem of the depiction of the number zero. No sign at all to stand for no thing is in itself no sign, that is, the lack of a sign. Thus the iconic representation of zero is paradoxical and self-negating. The only way out is to go beyond strictly iconic representations of number, which is what happens eventually, to all historical systems of numerals, or to broaden the range of icons employed. This is what the Mayans did, adopting a pictogram representing an empty turtle shell to represent zero ([Coe](https://en.wikipedia.org/wiki/Michael_D._Coe) 1987).

However, much better known and far more influential is Wittgenstein’s Meaning as Use theory. This lives up to its title, namely that the meaning of a word, sentence or other sign is given by its usage. How a sign is used in relation to other signs and human actions is what reveals its meaning, not some indication of objects beyond itself. Signs are used within language games which are a part of forms of life. Thus Wittgenstein’s (1953) account is based on a model of human beings as engaging in and living through Forms of Life, which can be interpreted as grounded social practices. Within Forms of Life, activities support the Language Games that are woven into them. Language Games are the human communicative practices that enable and regulate Forms of Life, but Language Games can also take on a ‘life of their own’ becoming internally driven, that is following the interests and desires of the participants. These may be detached from more grounded activities within Forms of Life, namely those that are tied into everyday functioning and survival. But no matter how elevated and abstract the language games become, such as is the case with mathematics, the human participants are embedded in other language games and forms of life that anchor them materially to matter and life.

It should be noted that the meaning of a sign is not given by a single instance of usage. One use can contribute to the meaning of the sign which is always growing as the pattern of the usage grows. Overall, Meaning as Use is an holistic theory, for meanings are not given explicitly, as when one can look up the meaning definition of a word in a dictionary. Rather it is the overall patterns of uses, the associated material activities in the form of life, and the nexus of connections and relationships into which the word, sentence or sign enters into, namely the history of its use, and expectations of its future uses that constitute its meaning. (McDermott 2001, Quine 1951)

The Meaning as Use theory applies to mathematics primarily via sign transformations and relationships with other signs. For such are the main usages of mathematics. One reading of this lies in Robert Brandom’s (2000) Inferentialism. This proposes that the meaning of a sign lies in the further signs it entails, and those that entail it. Thus, a mathematical sentence implies a whole range of other sentences and is itself implied by a further range of sentences. More generally a mathematical sign is enmeshed with a plethora of other signs, with an array of connected signs forming a web of meaning. In my view, restricting the signs to sentences and limiting the relations between them to logical deductions, as Inferentialism can sometimes appear to do, imposes too restricted a view of mathematical meaning. Mathematics contains a broader range of significant signs beyond sentences: there are also terms, concepts, diagrams as well as calculations, proofs, methods and theories. Likewise mathematics contains a wider range of important relationships beyond logical deduction including exemplification, instantiation, analogy, representation, translation, transformation, expansion (exposure of inner structure), induction and abduction (Peirce 1931-58). Without developing this vision any further, what can be claimed is that the meaning of a mathematical sign lies not just in the reference or direct designation of the sign, although this is not totally excluded, as it is one link. It is to be found in a network of relationships making up the web of meaning surrounding the sign. This links it to a system of other signs by a complex range of relationships. So the meaning of a sign is contextual, and holistic; it comprises the relationships with other signs (McDermott 2001, Quine 1951).

This is a complex theory requiring a much expanded treatment and more extensive justification. However I shall sidestep this by just focusing on the meaning of zero without developing this whole semantic and semiotic theory here.

Thus I can now say that the meaning of zero is not just that of a placeholder in a numerical array, although that is certainly part of its use. Nor does it lie solely in the analogy with Sunya or other concepts of emptiness, although the cardinality of any empty set is 0. Primarily, its meaning lies in its relationships within the domain of number. These relational linkages include (+1) - (+1) = 0, (+7) + (-7) = 0, (-107) + (+107) = 0, 7+0=7, 0x(-5) = 0, etc. Such relationships establish zero’s unique role as the identity for the operation of addition. It is Brahmagupta’s great contribution. Or rather, it is one of them, that he systematically laid out the properties and uses of zero within the systems of natural numbers and integers. By systematising these properties and uses of zero, as well as the relationships of positive and negative numbers and rules for using the four arithmetical operations, he has earned himself the epithet “The Euclid of Integer Arithmetic” which I previously suggested he deserves. The primary meaning of zero consists of its relationships and links with numbers and operations in the theory of integers, the true sentences that bind zero to positive and negative numbers via operations in equations.

It is important to remember that zero already has different sets of meanings in different contexts. Within the domain of number representation, namely numerals, zero serves as a placeholder, a sign indicating the equivalent of an empty abacus column in a place value numeral. In this sense, zero has not come into its more important full conceptual meaning. In the natural numbers zero serves as an additive identity but within a restricted domain of subtraction wherein a larger number cannot be subtracted from a smaller one. Within the integers zero fully fulfils its role an additive identity, because every integer has its own unique inverse. Thus for any integer *i*, there is an inverse element *-i* such that *+i* + *-i* = 0. Indeed subtraction is properly defined over the integers by *i* - *j* =def (*+i*) + (*-j*), the sum of a number (*i*) with another number’s (*j*) additive inverse. The meaning of zero changes still further as number systems expand to Q (the rational numbers), R (the real numbers) and finally C (the complex numbers). These expansions bring in more complications and change the meaning of zero, extending or changing its domain of use. For example, Q, the domain of rational numbers, is closed under division (the inverse of multiplication) except that division by zero is undefined (and not allowed). The same holds for R and Z.

*The Pragmatics of Mathematics*

The development of mathematics, arithmetic and indeed the concept and uses of zero all occur within the contexts of social historical practices. I have indicated how the needs of ancient societies in terms of taxes and trade shaped the origins of numeration and calculation technologies for its first two millennia of existence. Over the following three thousand years mathematics continued to develop, woven into various systems of modelling and control, including mensuration, geometry and trigonometry especially for the purposes of astronomical and astrological calculation, control and prediction. Royal courts, state institutions and religious settlements both trained skilled mathematicians and maintained them to fulfil these purposes.

To understand the social context of zero one must examine the Indian mathematical tradition that led up to Brahmagupta’s achievements. I believe that like Euclid, Brahmagupta pulled together centuries of work in mathematics to summarise and systematize. That tradition would have already had the concept and number zero, positive and negative numbers, the four arithmetical operations, place value numeration and much more mathematics including trigonometry, algebraic equations and solution methods, approximations for pi and other constants. Thus the huge achievement in defining the number zero and its properties and relationships belongs to a culture, a civilization, not just a single heroic figure, This is not to dethrone or denigrate Brahmagupta but to rebalance the history of mathematics and all human achievements as being social and group achievements, not those of outstanding individuals alone.

Isaac Newton acknowledged this when he wrote in a 1675 letter to fellow scientist Robert Hooke, “*If I have seen further, it is by standing on the shoulders of giants*”. In his day this was already a well-known phrase dating back to the twelfth century or earlier (Merton 1965). Thus Newton acknowledges his huge intellectual debt to his teachers and predecessors and recognises that his own contributions are only possible because of them. He is just one voice, albeit an important one, in the great conversation of humankind.

Joseph (1991) offers a great synoptic view of developments in Indian mathematics. In his Chronology of Indian History and Mathematics (pp. 313-314) he distinguishes a number of periods of development. In the period 500–200 BCE there is establishment of Indian states and the rise of Buddhism and Jainism. During this period the practice and development of Vedic mathematics continues but later declines with the ending of ritual sacrifices, with its need for mathematically precise altars. This is followed by the beginnings of Jain mathematics including number theory, permutations and combinations, the binomial theorem, and astronomical work.

During the next period that runs from 200 BCE to 400 CE there is a triple division of India, with the Kushan dynasty (North), Pandyas (South), Bactrian-Persian (Punjab). During this time Jain mathematics develops including rules of mathematical operations, decimal-place notation, and the first use of 0. Algebra emerges including simple, simultaneous, and quadratic equations. In addition square roots are treated as are details of how to represent unknown quantities and negative signs. Thus zero 0 and negative signs have already made their appearance, over two centuries before Brahmagupta.

The following period, one that incorporates the work of Brahmagupta, stretches from 400 CE to 1200 CE. This is the period of the Imperial Guptas, reaching their height in the reign of Harsha (606–647). This period sees the flowering of Indian civilization including developments across mathematics, science, philosophy, medicine, logic, grammar, and literature. In mathematics, this is regarded as the classical period or the golden age of Indian mathematics (Parameswaran 1998). Notable mathematicians of this period include Aryabhata I, Varahamihira, Bhaskara I, Brahmagupta, Sridhara, Mahavira, and Bhaskara II (also known as Bhaskaracharya). A whole string of important mathematical works were written, and in chronological order these are the *Bakshali Manuscript*, *Aryabhatiya*, *Panca-siddhantika*, *Aryabhatiya Bhasya*, *Maha Bhaskariya*, *Brahma Shputa-siddhanta* (Brahmagupta’s main mathematical work of 628 CE), *Patiganita*, *Ganita Sara Samgraha*, *Ganitilaka*, *Lilavati*, and *Bijaganita*. The mathematical advances across this period of 800 years are too extensive to list here and require a book length treatment on their own (see Joseph 2016).

Among the notable mathematicians listed here is Varahamihira (505–587). He was an important astrologer and mathematician, who among other things discovered a version of Pascal's triangle and worked on magic squares. He worked at one of the two leading mathematical centres in India in Ujjain, and Brahmagupta was its next major figure, so Varahamihira was a direct part of the tradition of learning that led to Brahmagupta (O'Connor and Robertson 2000a).

Even the contributions of Brahmagupta alone range across much of mathematics and include [Algebra](https://en.wikipedia.org/wiki/Brahmagupta#Algebra), [Arithmetic](https://en.wikipedia.org/wiki/Brahmagupta#Arithmetic) ([Series](https://en.wikipedia.org/wiki/Brahmagupta#Series) and [Zero](https://en.wikipedia.org/wiki/Brahmagupta#Zero)), [Diophantine analysis](https://en.wikipedia.org/wiki/Brahmagupta#Diophantine_analysis) ([Pythagorean triplets](https://en.wikipedia.org/wiki/Brahmagupta#Pythagorean_triplets) and [Pell's equation](https://en.wikipedia.org/wiki/Brahmagupta#Pell's_equation)), [Geometry](https://en.wikipedia.org/wiki/Brahmagupta#Geometry) ([Brahmagupta's formula](https://en.wikipedia.org/wiki/Brahmagupta#Brahmagupta's_formula), [Triangles](https://en.wikipedia.org/wiki/Brahmagupta#Triangles), [Brahmagupta's theorem](https://en.wikipedia.org/wiki/Brahmagupta#Brahmagupta's_theorem), [Pi](https://en.wikipedia.org/wiki/Brahmagupta#Pi) and [Measurements and constructions](https://en.wikipedia.org/wiki/Brahmagupta#Measurements_and_constructions)), [Trigonometry](https://en.wikipedia.org/wiki/Brahmagupta#Trigonometry) ([Sine table](https://en.wikipedia.org/wiki/Brahmagupta#Sine_table) and [Interpolation formula](https://en.wikipedia.org/wiki/Brahmagupta#Interpolation_formula)).

Brahmagupta lived and worked for a good part of his life in Bhillamala (modern [Bhinmal](https://en.wikipedia.org/wiki/Bhinmal) in [Rajasthan](https://en.wikipedia.org/wiki/Rajasthan), India) a centre of learning for mathematics and astronomy. Brahmagupta became an astronomer of the Brahmapaksha school, one of the four major schools of Indian astronomy during this period. He studied the five traditional *Siddhartha* on Indian astronomy as well as the work of other mathematicians and astronomers including [Aryabhata I](https://en.wikipedia.org/wiki/Aryabhata_I) and Varahamihira. In the year 628, at the age of 30, he composed the *Brahmasphutasiddhanta* (the improved treatise of Brahma) which is believed to be a revised version of the received *Siddhanta* of the Brahmapaksha school of astronomy. Scholars state that he incorporated a great deal of originality within his revision, adding a considerable amount of new material. The book consists of 24 chapters with 1008 verses in the [Arya metre](https://en.wikipedia.org/wiki/Arya_metre). A good deal of it is astronomy, but it also contains key chapters on mathematics, including algebra, geometry, trigonometry and algorithmics, which are believed to contain new insights due to Brahmagupta himself (Wikipedia 2021a).[[11]](#footnote-11)

What this abbreviated history shows is that first of all, Jain mathematics already includes the first use of 0 and the representation of negative signs by 400 CE, over two centuries before Brahmagupta. Second, by his own account Brahmagupta studied the *Siddhanta* text of the Brahmapaksha school of astronomy, and his own major contributionis the much revised and extended version of the received text. However, locating Brahmagupta in the Indian historical tradition which gave rise to him, and into which he himself contributed so much is by no means intended to belittle his contributions. Brahmagupta's understanding of the number systems went far beyond that of others of the period. In the *Brahmasphutasiddhanta* he defined zero as the result of subtracting a number from itself. He gave some further properties as follows:

*When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero.*

He also gives arithmetical rules in terms of fortunes (positive numbers) and debts (negative numbers):-

*A debt minus zero is a debt.  
A fortune minus zero is a fortune.  
Zero minus zero is a zero.  
A debt subtracted from zero is a fortune.  
A fortune subtracted from zero is a debt.  
The product of zero multiplied by a debt or fortune is zero.  
The product of zero multiplied by zero is zero.  
The product or quotient of two fortunes is one fortune.  
The product or quotient of two debts is one fortune.  
The product or quotient of a debt and a fortune is a debt.  
The product or quotient of a fortune and a debt is a debt.*

Brahmagupta then tried to extend arithmetic to include division by zero, Although we reject these latter suggestions, such as *Zero divided by zero is zero*, overall his work is a brilliant and groundbreaking attempt to extend arithmetic to negative numbers and zero (O'Connor and Robertson 2000b).

I locate Brahmagupta firmly within the Indian mathematical tradition of the golden age, rather than extolling him as an individual and heroic figure on his own. My aim is to critique the ‘great (hu)man theory’ of history. According to this, [history](https://en.wikipedia.org/wiki/Philosophy_of_history) can be largely explained by the impact of great humans, or [heroes](https://en.wikipedia.org/wiki/Hero); highly influential and unique individuals who, due to their natural attributes, such as superior intellect, have a decisive historical effect. The theory is primarily attributed to the Scottish philosopher and essayist [Thomas Carlyle](https://en.wikipedia.org/wiki/Thomas_Carlyle) (1841) who claimed that the history of the world is but the biography of great men.[[12]](#footnote-12) The ‘great human theory’ found many supporters in the Victorian era, including [William James](https://en.wikipedia.org/wiki/William_James). However, there was robust criticism of this view from Herbert Spencer (1873) who stated that “Before he can remake his society, his society must make him.” [Tolstoy](https://en.wikipedia.org/wiki/Tolstoy)'s [War and Peace](https://en.wikipedia.org/wiki/War_and_Peace) also features criticism of Great Human Theories as a recurring theme in the philosophical digressions. In the background first Hegel then Marx proposed that history is made by large scale forces that transcend individuals.

A little more recently, with respect to mathematics, Ogburn (1926, p. 227) argues explicitly against this theory stating, as an example, that “the discovery of the calculus was not dependent upon Newton, for if Newton had died, it would have been discovered by Leibnitz. And we think that if neither Leibnitz nor Newton had lived, that it would still have been discovered by some other mathematician.” Nevertheless, although *great human theories* have been widely rejected by historians in the modern era, they are still widespread in popular culture, unreformed school history teaching, and in some areas of leadership studies (e.g. Turak 2013).

Like Euclid did for geometry, Brahmagupta provides an explicit articulation and systematic synthesis of what was known about zero, negative and positive numbers and how all three types combine and interact through applications of the four arithmetical operations. As the author of a synthesis of the cultural knowledge of arithmetic it is even less appropriate to apply the great human theory of mathematical advances to Brahmagupta with respect to arithmetic than to some other mathematicians, who carved out whole new fields virtually on their own, although overall it is accepted that he personally made a great number of significant contributions to mathematics.

A number of questions about the emergence of zero remain. First, what was it within the culture of Indian civilization that led to the development of zero, or might one even say, necessitated its invention, with its full meaning and functions described here? Second, what is it that has made zero so difficult a concept and number, that prevented its invention in most cultures and delayed its transmission for so many centuries when it had been invented? Third, what further support is there for the account given here of the development of zero?

*What in Indian culture led to the development of zero?*

The first presence of zero as a half formed concept is as a place holder, a marker for an empty space in a compound numeral employing place value notation. Very large numbers were employed for a variety of purposes, mostly mythological, theological, astronomical and astrological, which provided an impetus for their development. These uses also led to a drive for optimum economy in the means of expressing number through an efficient and compressed system of numerals. In consequence the mark of an empty column, a dot, small circle, or a circle with a dot inside is used. It signifies an empty place within a larger compound numeral represent by a string of digits whose placements indicate their value or denomination. The long elaboration of the concepts of Sunya, sunyata meaning emptiness, nothing provides a philosophical backdrop for the emergence of the fully fledged concept of zero as a number.

*What made zero so difficult a concept?*

Numerous scholars have indicated the philosophical and theological difficulties in the network of interlinked concepts of Sunya, sunyata, emptiness, nothingness, nothing, non-being, absence, void, and so on. One of the fundamental problems is having a name or concept (present) for absence. Since the name is used as a marker for that which it names, that is, is used as if it was that which it names, a paradox or contradiction arises. For the name of nothing is not nothing but something. Unless a clear distinction is drawn between a realm of objects (existents), a language (collection of names, terms and sentences), and a metalanguage (a domain in which the names, terms and sentences of the language are themselves treated as objects) confusion is very easy. Thus a void is an emptiness, an absence in the domain of objects (level 0). But there is a word for void that is a linguistic term in language, and is existent in the domain of language and can be used in utterance in written language or speech. It is the 7th word in the previous sentence. However, the sign ‘void’ is an expression in the metalanguage (level 2) denoting the linguistic object, the word in between the quote marks (void), in the level 1 domain. In the metalanguage this word is mentioned, that is named, and the convention is that quote marks make the word (void) into the name of the word (‘void’).

|  |  |  |  |
| --- | --- | --- | --- |
| **Level** | **Domain** | **Range of reference** | **Example** |
| **Level 2** | **Metalanguage** | Linguistic constants (referencing specific linguistic terms and sentences) and linguistic variables (ranging across domains of linguistic terms and sentences) | The domain of signs that ranges over the first order (level 1) universe of language and other signs, for example, ‘void’, ’zero’, references to level 1 items such as all sentences defining nothing. Sentences applying non-existence to themselves are self-contradictory. |
| **Level 1** | **Language** | Names of specific objects, variables ranging across domain of objects (e.g., somebody who …), sentences describing relationships between objects | The universe of language and other signs for void, Sunya, sunyata, nothing, nothingness, non-being, absence, zero |
| **Level 0** | **Objects** | None. Objects are simply themselves without referential functions (unless treated as signs in another system, where they are *not* treated simply as objects) | Experience of the pure phenomenon of emptiness (void) |

*Table 4: Distinguishing the levels of objects, language and metalanguage*

The contradiction arises because an empty state of affairs, the void, the pure phenomenon of emptiness (on level 0), meaning that there is nothing present, is referred to by an object, nothing, which is an object in the domain of language (level 1). Furthermore, to provide this analysis I need to discuss or write about both level 0 objects, contents or states of affairs and level 1 signs which themselves refer to level 0 objects. For this I need the Level 2 metalanguage domain which can refer (write about) everything on lower levels. Level 1 enables me to *use* signs and language, but within a system with logically strict separation into levels, does not allow me to *mention (name or discuss)* level 1 signs.

Natural language such as I employ in writing this paper does not include level 0 objects, these exist outside of language and are only present at one remove, mediated via referral, through their names.[[13]](#footnote-13) In addition, natural language serves as its own metalanguage, so I can say that ‘Natural language serves as its own metalanguage’ is a sentence containing seven words. I can self-refer within this language. This means that level 2 collapses and everything is at level 1. This enables me to say that the term ‘void’ exists the same as any other term in the language, it is a normal word, but it names an emptiness, which is not the same as most linguistic objects, because its referent is non-existent. This is an antinomy or a paradox, making it hard to understand, and convincing some people it is unsound. But it is not a self-contradiction. To achieve the latter I can make the following statement. This very sentence you are reading now is itself false. (More briefly I can say: This sentence is false.) This is a self-contradiction. For if I assume that it is true, then by its own statement it is false. But if instead I assume that it is false, then it is not as it states false, but true. If we accept the Law of Excluded Middle, then two negatives make a positive, so it is both true and false at the same, making itself contradictory, and invalidating itself.

If I say, the void does not exist, this is not self-contradictory. Nor is ‘The void exists’. For it is possible that somewhere in the universes of matter or thought that there is a void or that it can be imagined, named or described. But if I say, non-existence exists, on the surface this is self-contradictory. Non-existence is a noun, and to exist is a verb. The sentence asserts that the state of non-being, that to which non-existence refers, has existence or being. This is a metaphysical or philosophical conundrum such as is found in Zen Buddhism and elsewhere. However, it is usually understood as referring to a human state of being, a sense of self or posture with regard to the world, or about the transcendence or lack of ego. It is not usually understood as meaning that there is nothing, in the sense that nothing at all exists in the universe. In my view, the latter is false empirically, because I can point to things that exist. Whether it is logically false too, because such a statement could not be true except in a universe with no consciousness, language or objects, making the articulation of such a sentence impossible, I am unsure.

Modern logic says that ‘non-existence exists’ is an illegitimate sentence because existence is not a predicate. Existence is not a property but is included in all quantification. One may say all, some, or none of some category have a given predicate or property. For example, black swans exist, meaning that there are some swans that are black. One can say nothing is material. That is, no material objects exist. Or everything is material. That is all objects are material. One can say the empty set exists. (There is an *X* such that *X* is a set, but there is no element *e* such that *e* belongs to *X*). However, in set theory one needs an axiom to make this sentence true. Even as late as 1888 the great mathematician Dedekind excluded the consideration of the empty set for reasons linked to problems of non-existence (Barton 2020).

Descartes’ ontological argument for the existence of god hinges on the use of existence as a predicate, and is rejected on this basis. Descartes’ argues that god is the perfect being, possessing all good attributes. Existence is an attribute, and evidently existence is a better attribute than non-existence. Therefore god has the better attribute of existence, that is, god exists.

One could continue, asking what the being of non-being means, if anything, and whether the non-being of being means the same, something different, or nothing at all. This plunges us into the domains of metaphysics or indeed wit. Many writers have played with the ambiguities of words that mean nothing or negate existence. In Homer’s Odyssey the hero Ulysses tells the Cyclops Polyphemus that his name is Nobody. Later when Ulysses has blinded Polyphemus the latter shouts out for help “Nobody is attacking me!”. His cries for help are self-defeating. Through this witty ambiguity Ulysses escapes.

Overall, it is clear that there are huge conceptual, philosophical and metaphysical difficulties in dealing with the concepts of void, emptiness, non-being, and so on (Barrow 2000). Some of these difficulties tainted the concept of zero and prevented its acceptance and spread. As well as philosophically difficult, naming the non-existent has been seen as blasphemous, which further helps to explain the obstacles confronting the concept, and delaying its acceptance and spread for centuries. Aristotle denied the existence of the void, and in the dark ages Christianity based its theology on his philosophy (Seife 2000). In medieval Europe right up until the time of Newton the idea of the void or the assertion of its existence was blasphemous because it negated the doctrine of the omnipresence of god (Barrow 2000).[[14]](#footnote-14)

*What support is there for this account of the development of zero?*

Perhaps the most authoritative accounts of the development of zero are given by the historian George Gheverghese Joseph. In a series of publications including Joseph (1991) Joseph (2008) and culminating in Joseph (2016), he has not only thoroughly researched and documented the origins of zero in India, but also looked at less developed precursors in all of the great civilisations as well as the parallel emergence of the full concept of zero in Mayan culture.

According to Joseph (2016), a number of factors led to the development of zero in Indian culture.

1. The invention and use of column based numerals with different denominations laid out in different columns, resulting in empty columns and later a sign representing an empty space in a column in compound numerals.
2. On the basis of (1) the development of a place value system for representing number with no drawn columns and a sign representing the lack of any figure the that particular denomination
3. The concept of void (Sunya) or emptiness as something significant, imaginable and representable
4. The use of numbers in astronomy, astrology and calendrical calculation as well as a fascination with large numbers for describing long periods of time
5. Cohorts of specialists including astronomers, astrologers, priests or scribes devoted to mathematics and calculation over a significant period of time

Not explicitly mentioned as a possible precursor of (1) is the use of an abacus or counting troughs to distinguish numeral denominations that may well have inspired the use of columns for recording number.

Joseph (2016) describes the occurrence of zero among the Mayans as manifested in the following.

1. A focus on extensive calendar computation including very large numbers requiring a place value numeral system
2. Groups of priests serving as teachers and writers including mathematical skills (possibly including a high ranking woman scribe)
3. A very economical place value numeral system initially only using 3 symbols (for 0, 1 and 5)

Much less is known about the Mayans than developments in India, hence what one can say about the origins and numeration and zero among the Mayans as well as their culture overall is very limited. Nevertheless, the comparison raises the question: is the concept of void a necessary conceptual prerequisite for the development of zero in its full sense? Did it occur in Mayan culture?

So the question is, did the Mayans have a concept of the void, which could be a precursor to zero? And the answer is yes, they did. The void is a part of their primordial world before anything exists, described in the opening of the Mayan sacred text Popol Vuh.

“This is the account of when all is still silent and placid. All is silent and calm. Hushed and empty is the womb of the sky. … There is not yet anything that might exist.” (Christenson 2007, p. 58). Like in many creation myths out of nothing emerges something, the empty void is pregnant, and spews forth all in the world and heavens that comes to exist. This provides confirmation for the conjecture that having the concept of the void is a necessary prerequisite for zero.

Interestingly, in comparison, Ancient Egypt does not seem to have a concept of an empty void. Their creation myths begin with nothing but endless dark water without form or purpose.[[15]](#footnote-15) All of creation emerges from this source, but it is not a void (Wikipedia 2021c). They did have a hieroglyph that served some of the functions of zero, also standing for beauty and completeness. Its consonant sounds were ‘nfr’ (vowels unknown).This symbol was used to express nil remainders in a monthly account sheet from the Middle Kingdom dynasty 13 (c. 1770 BCE). The bookkeeping record looks like a double entry account sheet with separate columns for each type of goods. At the end of the month, the account was balanced. When the total income matched the total disbursement this was shown by the nfr symbol (Lumpkin 1997). One would like to think that this means income minus disbursement equals zero. But it does not. What it probably means is that the books are balanced, the results are complete and beautiful. Perhaps lacking the concept of void kept the Egyptians from inventing zero.[[16]](#footnote-16)

**Conclusion**

I began by criticising Seife’s (2000) blurring of the distinction between the concepts of nothing, emptiness, the void and the quantitative mathematical concept of zero. Likewise I pointed to the dangers of identifying the concepts of fullness, unity and integrity with the mathematical *one*. It is clear that the history of mathematics is bedevilled with such confusions and they may have delayed the acceptance of zero as a number by a thousand years in the West. Other civilizations were not so tardy in adopting it. But Seife is right to say 1 comes before 0 in historical development, and indeed it is the same in the cognitive development (Barton 2020). From one to zero in terms of mathematical understanding took millennia.

The only other ‘number’ so bedevilled is that of infinity. Indeed a controversy is currently raging over whether infinity repackaged as the concept Grossone can be introduced into numerical calculations (Mathoverflow 2020, Calude and Dumitrescu 2020). However, tracing the problems and the history and philosophy of infinity is a whole other quest, taking us away from zero. Despite its former associations, zero is now an unproblematic concept within mathematics.

Perhaps I should not be condemning of Seife because the urge to play with the different cultural meanings of zero and one is irresistible. Malcolm Forbes wrote "Education's purpose is to replace an empty mind with an open one." (RightAttitudes 2021). What is this but the story of *zero* to *one*?

In this paper I have traced the mathematical and philosophical significance of zero, as the title promises. But also in my title I said Nought Matters. Perhaps the meaning of title is obvious but let me spell it out. One of the amazing and difficult things about zero, nothing, etc., is the ambiguity, paradoxicality, the conundrums and indeed humour that emerge from its study and its self-negation (which impacted on its history mightily). I chose 'Nought Matters' because it plays on three meanings –

First, the synonymous sentence: ‘nothing matters’, means it is not the case that anything has significance. This says, jokily, that my enterprise (and everything) has no point

Second, I can put this as another synonymous sentence: ‘zero counts’, meaning zero and its synonyms refer to contents of importance. This surely is my message. (But it also plays with the ambiguity of the word ‘counts’ meaning both to enumerate, and to be of importance. A zero count can also be the result of a process of enumeration that ends as it starts returning the number zero as the outcome.)

Third, as a term it is synonymous with the noun phrase ‘nothing matters’, describing matters or materials relating to nothing or zero.

This last meaning, namely matters concerning zero, describes what comes from a study of zero, and the issues it raises. It encapsulates the subject of this paper and indeed the whole conference and project of ZerOrigIndia.

Whitehead (1911, p. 63) said: “The point about zero is that we do not need to use it in the operations of daily life. No one goes out to buy zero fish. It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought.” After the initial observation, he is of course wrong, at least today. No one in modern society can go a day without using zero. Even if they do not look at a clock or thermometer display, do not read a paper or a book, do not go shopping or handle any money, as soon as they look at any electronic display, use a mobile phone, tablet, computer, radio or television, travel in a car, bus, ship or plane, they are seeing the results of millions upon millions of zeros (and ones) controlling the system and encoding data. Zero is everywhere, throughout human society. I’m tempted to say zero or nothing means everything. But that would contradict my attempts to keep the quantitative and qualitative aspects of zero separate.

Zero is the only essential and ineliminable number. Even 1, the only other possible candidate for this title, is derived from 0 as its successor. Thus despite, or perhaps because of the controversy its adoption has caused over 3000 years, zero with its many functions and roles is probably the most important of all of the numbers. Certainly the modern industrial revolution based on information, communication and control technologies could not even begin without it. Perhaps too the original industrial revolution depended on it. So much rests on zero and the number systems of which it is the foundation.

In coming to the end of my voyage of discovery it behoves me to give the last word to the land of its origin. Pujyam is the word for zero in Tamil, Malayalam and Marathi, and it means ‘worthy of worship’. As we have seen, history confirms this judgement. Beyond this, Sarma (2011, p. 211) reports a Tamil proverb that declares “Inside the pujyam (zero), there exists a rajyam (kingdom)”. I hope that I have opened the door to this kingdom a crack and offered a brief look inside.

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Note that a slightly abbreviated version of this paper is published as Ernest (2024), in the marvellous compendium on zero: P. [Gobets](https://brill.com/search?f_0=author&q_0=Peter+Gobets) and R. L. [Kuhn](https://brill.com/search?f_0=author&q_0=Robert+Lawrence+Kuhn), Eds. (2024) *The Origin and Significance of Zero - An Interdisciplinary Perspective*. Leiden, The Netherlands: Brill.

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1. Suppose to the contrary, that Sn increases the value of n more than Sm increases that of m. Then we have an immediate contradiction, because this can be expressed as S*n* - *n* > S*m* - *m*, that is *n*+1-*n* > *m*+1-*m*, and hence 1>1, using the normal arithmetic operations as defined in Peano Arithmetic. [↑](#footnote-ref-1)
2. An irony is that although measures such as use of a graduated ruler are excluded by pure Euclidean geometry, and regarded as inferior by the purist ideologies of Ancient Greek and 19th century pure mathematicians (Ernest 2020a), pure, notional measures along the coordinates of Cartesian geometry are admitted and much lauded within the hallowed domain of pure mathematics. [↑](#footnote-ref-2)
3. I leave aside the secondary impacts arising from the want and need to observe, record and predict astronomical, astrological and calendrical phenomena. [↑](#footnote-ref-3)
4. Negative numbers appear for the first time in recorded history in the [*Nine Chapters on the Mathematical Art*](https://en.wikipedia.org/wiki/Nine_Chapters_on_the_Mathematical_Art) (Jiu zhang suan-shu), which in its present form dates from the period of the [Han Dynasty](https://en.wikipedia.org/wiki/Han_Dynasty) (202 BCE – CE 220), (Wikipedia 2021b) and in Jain mathematics, as I record subsequently. [↑](#footnote-ref-4)
5. There is an important difference between a negative number, such as -7 and the operation of subtracting 7, even though we may incorrectly refer to both as ‘minus seven’. Historically, the emergence of subtraction preceded the recognition of negative numbers. Formally we would define the operation of minus seven as the addition of -7, even though this is an inversion of the actual order of their historical emergence. [↑](#footnote-ref-5)
6. Interestingly, Bronkhorst (2001) also makes a comparison with Euclid. He compares Panini’s systematic linguistics and grammar with Euclid’s elements, not Brahmagupta’s contribution as I do here. Of course there is no reason why one such comparison should invalidate the other. [↑](#footnote-ref-6)
7. Zephirus is also the origin of the Cabbalistic sephiroth, the tree of life. The creation myth for this begins with Ayin (zero, the void), from which comes Ayin Sof (God) who gives off an emanation of golden light Ayin Sof Aur that brings the tree of life into being, with its ten nodes corresponding to the numbers 1 to 10. Thus, the Cabbala, which at its very heart is numerological, is based on a creation myth that starting from 0 creates the numbers 1 to 10, and through them, creates the whole universe, seen and unseen. [↑](#footnote-ref-7)
8. As I sit here at my computer imperceptible to me signals carrying hundreds of radio and TV channels are passing through or around my body as are further Wi-Fi, mobile phone and Bluetooth communications. The empty void around me contains not only air but these mysterious forms of radiation that would have been dismissed as devilish emanations or mystical clap-trap a couple of centuries ago, alongside rumours of invisibly tiny malevolent creatures we now know as bacteria and viruses. [↑](#footnote-ref-8)
9. An obvious and central difference I should mention is that Formula 1 describes a property discovered about numbers, that they are uniquely decomposable as a product of primes. This much more amazing than Formula 2 which is simply the definition of the base ten place value representation of numbers, albeit developed over millennia and rarely ever fully analysed as here. By the time you can understand Formula 2 you are long past leaning how to fully use base ten place value representations of numbers, so it is in a sense, superfluous and regarded as mathematically trivial. But it is one of the most important conventions in all of mathematics, especially with respect to its applications. [↑](#footnote-ref-9)
10. Mathematical language is performative in two ways, which I term inner and outer. What I describe here is the inner performativity, whereby mathematical sign usage creates mathematical objects. The outer performativity of mathematics is the way it formats the way we experience and interact with the material world of (Skovsmose 2019, 2020, Ernest 2019). [↑](#footnote-ref-10)
11. In this paragraph and elsewhere I have replaced inflected letters by plain alphabetic letters to simplify the presentation, as scholars employing the alphabetisation of Sanskrit and other Indian written scripts will notice. [↑](#footnote-ref-11)
12. This brief reference should not be taken as an overall dismissal of Carlyle’s work, and there are more nuanced and positive evaluations of it (Baily 2006, Eliot 1855) [↑](#footnote-ref-12)
13. This differs from mathematical language, as I explained above, where terms and sentences (level 1) can become their own referents, the very objects and relationships that they name (level 0), through their (inner) performativity. This alchemy occurs because mathematical objects are primarily processes (applied to mathematical objects) that become reified and objectified analogously to the way verbs are transformed into nouns in nominalisation. For example, the actions of counting become the numbers themselves. This is very difficult to describe because the meanings are not static but change during this process. [↑](#footnote-ref-13)
14. To this day, nothingness, if not zero, is regarded by some as evil. “Evil is nothingness. ‘Evil’ is not *defined* as nothingness by Barth. Rather, evil is *identified* by Barth as *nothingness*.” (Wolterstorff 1996, p. 585. Original italics) [↑](#footnote-ref-14)
15. This may be because of the overwhelming importance of the Nile waters for their culture. [↑](#footnote-ref-15)
16. In a revised shorter version of this paper (in Ernest, P. (2024)’ based on new evidence from the Egyptologist F. Hoffman, I corrected my remarks on Egypt and zero as follows.

    In comparison, Ancient Egyptian culture included the idea of ‘that which does not exist’ in opposition to ‘that which exists’, according to Hoffmann (2021). Thus, the Ancient Egyptians had the concept of the Void. However, they lacked the concept of zero in their arithmetic.

    The Ancient Egyptians did have a hieroglyph that served some of the functions of zero, also standing for beauty and completeness. Its consonant sounds were ‘nfr’ (vowels unknown). This symbol was used to express nil remainders in a monthly account sheet from the Middle Kingdom dynasty 13 (c. 1770 BCE). The bookkeeping record looks like a double entry accounts sheet with separate columns for each type of goods. At the end of the month, the account was balanced. When the total income matched the total disbursement this was shown by the nfr symbol (Lumpkin, 1997). One would like to think that this means income minus disbursement equals zero. But it does not mean zero (although, according to Hoffmann, some Egyptologists disagree.) Assuming that such an interpretation is incorrect, it might be described as *presentism*, seeing the past through modern eyes, or even wishful thinking. What the use of *nfr* very likely means is that the books are balanced, the results are complete, symmetrical, and beautiful.

    Nevertheless, the Ancient Egyptians computed perfectly well without Zero (as also did the Mesopotamians and Ancient Greeks, for example.) So, the concept of Void is at best Necessary but appears not to be Sufficient for an advanced civilization to develop the mathematical concept of Zero. [↑](#footnote-ref-16)